Q. Code: P116 Reg. No. .... Name:........................

MSc DEGREE END SEMESTER EXAMINATION 2014–15
SEMESTER-1: MATHEMATICS
COURSE CODE - P1MATTO2: BASIC TOPOLOGY

Time: 3 Hours Max. Marks: 75

Part A

Answer any 5 Questions. Each carries 2 marks

1. State true or false:
   (a) *A subspace of a metric space is metrizable*
   (b) *Any discrete space is second countable.*
2. Prove that $\mathscr{C} = \{[a,b] \,:\, a,b \in \mathbb{R}, a < b\}$ cannot be a base for any topology on $\mathbb{R}$.
3. (a) Define second countable space?
   (b) Prove or disprove: $\mathbb{R}\times\mathbb{R}$ *is second countable.*
4. Prove that $\overline{A\cup B} = \overline{A}\cup\overline{B}$
5. Give an example of a connected closed subset C of $\mathbb{R}^2$ such that $\mathbb{R}^2 - C$ has infinitely many components.
6. State the Lebesgue Covering Lemma.
7. Give an example of a T_1-space which is not T_2. Justify?
8. Show that $\mathbb{R}$ is homeomorphic to $\mathbb{R}\times\{1\} = \{(x,1) \,:\, x\in\mathbb{R}\}$.

$2\times 5 = 10$

Part B

Answer any 5 Questions. Each carries 5 marks

9. Let d be the usual metric on $\mathbb{R}$. Show that $U\subset\mathbb{R}$ is an open ball if, and only if U is an open interval in $\mathbb{R}$.
10. Let $f : X\to Y$ be a continuous function. The graph of f is defined to be the set
$$G = \{(x, f(x)) \,:\, x\in X\}.$$
Then G is a subspace of $X\times Y$ with product topology. Prove that G is homeomorphic to X.
11. Let $(X,\mathcal{T})$ be a topological space and $A\subset X$. Then show that A is compact subset of X if, and only if the subspace $(A, \mathcal{T}|_A)$ is compact.
12. Prove that a closed subspace of a Lindelöff space is Lindelöff.
13. Prove that the unit circle S^1 is compact.

14. Prove or disprove: *In a T_1-space, limits of sequences are unique.*

15. Show that a continuous bijection from a compact space on to a Hausdorff space is a homeomorphism.

16. Show that every regular, second countable space is normal.

$5 \times 5 = 25$

Part C

Answer either (a) or (b) of the following four questions. Each carries 10 marks

17. (a) Let X be a non-empty finite set containing n elements and $T(n)$ be the number of topologies defined on X. Prove that for $n > 1$

$$2^n \leq T(n) \leq 2^{2^n - 2}$$

(b) Let X be a set and $\mathscr{P}(X)$ denotes the power set of X. Define $\theta : \mathscr{P}(X) \to \mathscr{P}(X)$ by

$$\theta(A) = A \text{ for all } A \in \mathscr{P}(X).$$

i. Show that θ satisfies the closure axioms.

ii. Find the corresponding topology on X.

18. (a) Let S^1 be the unit cirlcle in $\mathbb{R}^2$. Obtain S^1 as a quotient space of $[0,1]$.

(b) i. Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ are topological spaces and $f : X \to Y$. Define the weak topology on X determined by f.

ii. Let $X = X_1 \times X_2 \times \cdots \times X_n$, where $(X_i, \mathcal{T}_i), i = 1, 2, \ldots, n$ are topological spaces. Show that the product toplogy on X is the weak toplogy determined by the projection functions $\pi_i : X \to X_i$.

19. (a) i. Show that every path connected space is connected.

ii. Show by an example that the converse of the above statement need n't be true.

(b) i. Show that the set of all rational numbers as a subspace of $\mathbb{R}$ with usual topology is not connected.

ii. Show that a subset of $\mathbb{R}$ is connected if, and only if it is an interval.

20. (a) i. Prove the following (Wallace's Theorem)
Let A, B be compact subsets of topological spaces X, Y respectively. Let W be an open subset of $X \times Y$ containing the rectangle $A \times B$. Then there exist open sets U, V in X, Y respectively such that $A \subset U, B \subset V$ and $U \times V \subset W$.

ii. Using Wallace's theorem prove that every compact Hausdorff space is T_4.

(b) i. Show that there can be no continuous one-to-one map from the unit circle S^1 into the real line $\mathbb{R}$.

ii. Suppose $(X, \mathcal{T}_1)$ is a compact space and $(X, \mathcal{T}_2)$ is a Hausdorff space. If $\mathcal{T}_1 \supset \mathcal{T}_2$ show that $\mathcal{T}_1 = \mathcal{T}_2$.

$10 \times 4 = 40$