Reg. No .... Name **20P3053**

M. Sc DEGREE END SEMESTER EXAMINATION - OCT/NOV 2020: JAN 2021

SEMESTER 3 : MATHEMATICS

COURSE : **16P3MATT15 : NUMBER THEORY**

(For Regular - 2019 Admission and Supplementary - 2016/2017/2018 Admissions)

Time : Three Hours Max. Marks: 75

PART A

Answer all (1.5 marks each)

1. Prove that $\varphi(n)$ is even for $n \geq 3$
2. Suppose that the set $\mathscr{F}$ of all arithmetical functions is closed under Dirichlet multiplication. Prove that there is an identity element in $\mathscr{F}$.
3. Assume $(a, m) = d$. Prove that the linear congruence $ax \equiv b(\mod m)$ has solutions, if and only if, $d|b$.
4. Let $(a, m) = 1$. Prove that $ax \equiv b(\mod m) \implies x \equiv ba^{\varphi(m)-1}(\mod m)$.
5. Define proper factorization in a commutative ring with unity.
6. Let D be a domain. Prove that any two units are associates and any associate of a unit is a unit.
7. Let D be a domain and x and y non-zero elements of D. Prove that x is irreducible if and only if $\langle x\rangle$ is maximal among the proper principal ideals of D.
8. If $\mathfrak{a} \neq 0$ is an ideal of $\mathfrak{O}$ with $\mathrm{N}(\mathfrak{a})$ is prime , prove that $\mathfrak{a}$ is prime.
9. Prove that every non-zero ideal of $\mathfrak{O}$ has a finite number of divisors.
10. Prove that $\mathbb{R}[x, y]/\langle x\rangle$ is isomorphic(as rings) to $\mathbb{R}[y]$.

(1.5 x 10 = 15)

PART B

Answer any 4 (5 marks each)

11. Prove that two lattice points (a, b) and (m, n) are mutually visible if, and only if, $a - m$ and $b - n$ are relatievly prime.
12. Prove that the set of lattice points visible from the origin has density $6/\pi^2$
13. Prove that for all $x \geq 1$, $\sum_{n\leq x} \sigma_1(n) = \frac{1}{2}\zeta(2)x^2 + O(x \log x)$.
14. Prove that for $n \geq 1$, $\frac{1}{6} n \log n < p_n < 12 \left(n \log n + n \log\left(\frac{12}{e}\right)\right)$ where p_n is the n^{th} prime.
15. Prove that the units $U(R)$ of a commutative ring R with unity form a group under multiplication.
16. Prove that every non zero prime ideal of $\mathfrak{O}$ is maximal.

(5 x 4 = 20)

PART C

Answer any 4 (10 marks each)

17.1. Prove that if $x \geq 1$ and $\alpha > 0, \alpha \neq 1 \sum_{n \leq x} \sigma_\alpha(n) = \frac{\zeta(\alpha+1)}{\alpha+1} x^{\alpha+1} + O(x^\beta)$ where $\beta = \max\{1, \alpha\}$.

OR

2. Prove that $\sum_{p \leq x} \left[\frac{x}{p}\right] \log p = x \log x + O(x)$ for $x \geq 2$ where the sum is extended over all primes $\leq x$.

18.1. 1. Prove the converse of Wilson's theorem.
2. Find all positive integers n for which $(n-1)! + 1$ is a power of n.

OR

2. If p is odd, $p > 1$, prove that

1. $1^2 3^2 5^2 \ldots (p-2)^2 = (-1)^{P+1)/2} (\mod p)$
2. $2^2 4^2 6^2 \ldots (p-1)^2 = (-1)^{(P+1)/2} (\mod p)$.

19.1. Prove that a prime in a domain D is always irreducible. Is converse true? Justify.

OR

2. Define Euclidean quadratic Field. Prove that the ring of integers $\mathfrak{O}$ of $\mathbb{Q}(\sqrt{d})$ is Euclidean for $d = -2, -11$.

20.1. Prove that the non-zero fractional ideals of $\mathfrak{O}$ form an abelian group under multiplication.

OR

2. Prove that every non-zero ideal of $\mathfrak{O}$ can be written as a product of prime ideals, uniquely up to order of the factors.

(10 x 4 = 40)