Reg. No .... Name **20P1031**

M. Sc DEGREE END SEMESTER EXAMINATION - OCT 2020 : FEBRUARY 2021

SEMESTER 1 : MATHEMATICS

COURSE : **16P1MATT03 ; MEASURE THEORY AND INTEGRATION**

(For Regular - 2020 Admission and Supplementary 2016/2017/2018/2019 Admissions)

Time : Three Hours Max. Marks: 75

PART A

Answer All (1.5 marks each)

1. Prove that [0,1] is uncountable.
2. Give an example of a non-measurable function.
3. Let $E \subset M$ and M be measurable with $m(M) < \infty$. If E is measurable, show that

$$m(M) = m^*E + m^*(M - E).$$

4. Give an example of a Lebesgue measurable function which is not Riemann integrable.
5. Define canonical representation of a simple function.
6. If ϕ is a non -negative simple function and $A \supset B$, then prove that $\int_A \phi \geq \int_B \phi$.
7. Let (X, B, μ) be a measure space and f be a non-negative measurable function defined on X. Prove that the set function ϕ defined as B by $\phi(E) = \int_E f d\mu$ is a measure.
8. Let X be any uncountable set and $\mathcal{B}$ be the family of all subsets E of X, which are either countable or the complement of a countable set. Prove that $\mathcal{B}$ is a σ- algebra of subsets of X.
9. Let $(X, \mathcal{B}, \mu)$ be a measure space and $Y \in \mathcal{B}$.
Let $\mathcal{B}_Y$ consists of those sets of $\mathcal{B}$ that are contained in Y.
Set $\mu_{_Y} E = \mu E$ if $E \in \mathcal{B}_Y$.
Then prove that $(Y, \mathcal{B}_Y, \mu_{_Y})$ is a measure space.
10. If $V \subset X \times Y$, then prove that $(\chi_{_V})_x = \chi_{_{V_x}}$ and $(\chi_{_V})^y = \chi_{_{V^y}}$.

(1.5 x 10 = 15)

PART B

Answer any 4 (5 marks each)

11. (a) If E_1 and E_2 are two measurable sets, then prove that $E_1 \cup E_2$ is measurable.
(b) If E is a measurable set, prove that $\tilde{E}$ is measurable. Deduce that $E_1 \cap E_2$ and $E_1 \triangle E_2$ are also measurable.
12. For $k > 0$ and $A \subset R$, let $kA = \{kx : x \in A\}$ show that,
(i) $m^*(kA) = km^*A$ and
(ii) A is measurable if and only if kA is measurable.
13. Let $\langle u_n \rangle$ be a sequence of non-negative measurable functions and let $f = \sum\limits_1^\infty u_n$. Then prove that $\int f = \sum\limits_1^\infty \int u_n$.
14. Let f and g be integrable over E. Then prove that
(a) The function cf is integrable over E and $\int_E cf = c \int_E f$ (c is a constant)
(b) The function $f + g$ is integrable over E and

$$\int_E (f + g) = \int_E f + \int_E g.$$

15. (a) If $\langle E_i \rangle$ is a sequence of sets in $\mathcal{B}$, where $(X, \mathcal{B}, \mu)$ is a measure space, then prove that

$$\mu(\bigcup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} \mu E_i$$

(b) Let $\{A_n\}$ be a countable collection of measurable sets. Then prove that

$$\mu(\bigcup_{k=1}^{\infty} A_k) = \lim_{n\to\infty} \mu(\bigcup_{k=1}^{n} A_k).$$

16. Prove that $\mathcal{S} \times \mathcal{J} = \mathcal{S}(\mathcal{E})$, the σ-algebra generated by $\mathcal{E}$.

(5 x 4 = 20)

PART C
Answer any 4 (10 marks each)

17.1. (a) Prove that the collection $\mathcal{M}$ of all measurable sets is a σ-algebra.
(b) Prove that (a, ∞) is measurable for all $a \in R$.

OR

2. (a) If f is a measurable function, then prove that $\mathcal{M} = \{E : f^{-1}(E) \text{ is measurable}\}$ is a σ-algebra.
(b) If B is a Borel set, prove that $f^{-1}(B)$ is measurable.
(c) If $\langle f_n \rangle$ is a sequence of measurable functions (with the same domain), then prove that
(i) $\sup\{f_1, f_2, \ldots, f_n\}$ is measurable.
(ii) $\sup_n f_n$ is measurable.
(iii) $\overline{\lim} f_n$ is measurable.

18.1. (a) Define Riemann integral of a bounded function over a finite closed integral $[a, b]$ interms of step functions.
(b) Define Lebesgue integral of a bounded measurable function defined on a measurable set E with mE finite.
(c) Let f be a bounded function defined an $[a, b]$. If f is Riemann integrable, then prove that it is measurable and

$$R\int_a^b f(x)dx = \int_a^b f(x)dx.$$

OR

2. (a) Let f be defined and bounded on a measurable E with mE finite. Prove that $\inf_{f\leq\psi} \int_E \psi(x)dx = \sup_{\varphi\leq f} \int_E \phi(x)dx$ for all simple functions ϕ and ψ if and only if f is measurable.
(b) Using (a) give the definition of Lebesgue integral of a bounded measurable function over a measurable set E with mE finite.

19.1. (a) Let f be an extended real valued function defined on X, where $(X, \mathcal{B})$ is a measurable space. Then prove that
the following statements are equivalent:
(i) $\{x \in X : f(x) < \alpha\} \in \mathcal{B}$ for each $\alpha \in R$
(ii) $\{x \in X : f(x) \leq \alpha\} \in \mathcal{B}$ for each $\alpha \in R$
(iii) $\{x \in X : f(x) > \alpha\} \in \mathcal{B}$ for each $\alpha \in R$
(iv) $\{x \in X : f(x) \geq \alpha\} \in \mathcal{B}$ for each $\alpha \in R$
(b) If μ is a complete measure and f is a measurable function, then prove that $f = g$ a.e. implies g is measurable.

OR

2. (a) State and prove Jordan decomposition theorem.
(b) Let E be a measurable set such that $0 < \nu E < \infty$. Then prove that there is a positive set $A \subset E$ with $\nu A > 0$.

20.1. If $\mathcal{A}$ is an algebra, then prove that

$$S(\mathcal{A}) = \mathcal{M}_{\circ}(\mathcal{A})$$

OR

2. Let $[[X, \mathcal{S}, \mu]]$ and $[[Y, \mathcal{J}, v]]$ be σ-finite measure spaces. For $V \in \mathcal{S} \times \mathcal{J}$, write $\phi(x) = \nu(V_x)$ and $\psi(y) = \mu(V^y)$ for all $x \in X$ and $y \in Y$. Then prove that ϕ is $\mathcal{S}$-measurable and ψ is $\mathcal{J}$-measurable and $\int_X \phi d\mu = \int_Y \psi d\nu$.

(10 x 4 = 40)