Reg. No .... Name **20P3018**

M. Sc DEGREE END SEMESTER EXAMINATION - OCT/NOV 2020: JAN 2021

SEMESTER 3 : MATHEMATICS

COURSE : **16P3MATT12 : ADVANCED FUNCTIONAL ANALYSIS**

(For Regular - 2019 Admission and Supplementary 2016/2017/2018 Admissions)

Time : Three Hours Max. Marks: 75

PART A

Answer all (1.5 marks each)

1. We know a functional is a particular case of an operator. There can exist three types of convergence of a sequence of operators. Why there exist only two types of convergence of a sequence of functionals?.
2. If $T : X \to X$ is a contraction on a metric space X, show that T^n $(n \in N)$ is a contraction on X.
3. In a Hilbert space H, prove that $x_n \overset{w}{\to} x$ if and only if $\langle x_n, z\rangle \to \langle x, z\rangle$ for all $z \in H$.
4. Define the following terms:-
   (a) Domain in the complex plane. (b) Holomorphic function of a complex variable.
5. Show that the set of all bounded linear operators on a vector space into itself forms an algebra.
6. Let $T : X \to Y$ be a linear operator where X and Y are normed spaces. If T is bounded and $\dim R(T) < \infty$ then prove that T is compact.
7. Prove that every bounded linear operator $T : X \to Y$, where X and Y are normed spaces of finite rank is compact.
8. If $T \geq 0$, then show that $(I + T)^{-1}$ exists.
9. Find the operators $T : R^2 \to R^2$ such that $T^2 = I$. Write the positive square root of I.
10. Let $T : H \to H$ be a bounded self adjoint linear operator on a finite dimensional complex Hilbert space H. If T is represented by a diagonal matrix, show that the matrix is real. What is the spectrum of T?

(1.5 x 10 = 15)

PART B

Answer any 4 (5 marks each)

11. (a) Prove that every strongly operator convergent sequence is weakly operator convergent.
    (b) Is the converse of (a) true?. Justify.
12. Let $T_n \in B(X, Y)$, where X is a Banach space and Y a normed space. If (T_n) is strongly operator convergent with limit T, then prove that $T \in B(X, Y)$. If X is only a normed space (ie., X is not complete), will T be bounded always?. Justify.
13. Give an example of a linear operator $T : l^2 \to l^2$ whose residual spectrum is non-empty. Explain?.
14. Let A be a complex Banach Algebra with identity e. Then for any $x \in A$ prove that $\sigma(x) \neq \phi$.
15. Prove that every relatively compact set in a metric space is totally bounded.
16. Prove that any bounded self adjoint linear operator $T : H \to H$ on a Hilbert space $H, ||T|| = \sup\limits_{||x||=1} |\langle Tx, x\rangle|$.

(5 x 4 = 20)

PART C
Answer any 4 (10 marks each)

17.1. (a) Prove that a sequence (f_n) of bounded linear functionals on a Banach space X is weak * convergent if and only if

(i) The sequence $(||f_n||)$ is bounded.

(ii) The sequence $(f_n(x))$ is Cauchy for every x in a total subset M of X.

(b) Let $T : \mathcal{D}(T) \to Y$ be a linear operator where $\mathcal{D}(T) \subset X$ and X and Y are normed spaces. Then prove that T is closed if and only if it has the following property ``if $x_n \to x$, where $x_n \in \mathcal{D}(T)$ and $Tx_n \to y$, then $x \in \mathcal{D}(T)$ and $Tx = y$''.

OR

2. (a) Let $T : \mathcal{D}(T) \to Y$ be a bounded linear operator where $\mathcal{D}(T) \subset X$ and X and Y are normed spaces. Then prove the following.(i) If $\mathcal{D}(T)$ is a closed subset of X, then T is closed.
(ii) If T is closed and Y is complete, then $\mathcal{D}(T)$ is a closed subset of X.

(b) Let T be a closed linear operator from a Banach space X into a normed space Y. If T^{-1} exists and T^{-1} is bounded, then prove that $R(T)$ is a closed subset of Y.

18.1. State and prove spectral mapping theorem for polynomials

OR

2. (a) Let $T : X \to X$ be a bounded linear operator on a complex Banach space X.
Prove that for any $\lambda_0 \in \rho(T)$, $R_\lambda(T)$ has the representation
$R_\lambda(T) = \sum_{j=0}^{\infty} (\lambda - \lambda_0)^j R_{\lambda_0}(T)^{j+1}$ and the series is absolutely convergent within the open disc given by $|\lambda - \lambda_0| < \frac{1}{||R_{\lambda_0}(T)||}$

(b) Let $T \in B(X, X)$ where X is a complex Banach space. If $\lambda, \mu \in \rho(T)$, prove the following

(i) $R_\mu - R_\lambda = (\mu - \lambda) R_\mu R_\lambda$

(ii) R_λ commutes with any $S \in B(X, X)$ satisfying $ST = TS$

(iii) $R_\lambda R_\mu = R_\mu R_\lambda$.

19.1. Let $T : X \to X$ be a compact linear operator on a normed space X. Then prove that for any $\lambda \neq 0$, the range of $T_\lambda = T - \lambda I$ is closed.

OR

2. Let $T : X \to X$ be a compact linear operator on a normed space X and let $\lambda \neq 0$. Then prove that there exists a smallest integer r (depending on λ) such that from $n = r$ on the null spaces $N(T_\lambda^n)$ are all equal and if $r > 0$, the inclusion
$N(T_\lambda^0) \subset N(T_\lambda) \subset N(T_\lambda^2) \subset \ldots \subset N(T_\lambda^r)$ are all proper.

20.1. (a) Define a monotone sequence (T_n) of bounded self adjoint linear operators $T_n : H \to H$ on a complex Hilbert space H.

(b) Let (T_n) be a sequence of bounded self adjoint linear operators on a complex Hilbert space H such that $T_1 \leq T_2 \leq T_3 \leq \ldots \leq K$. Suppose that T_j commutes with K and with every T_m. Then prove that (T_n) is strongly operator convergent. $(T_n x \to Tx$ for all $x \in H)$ and the limit operator T is a bounded self adjoint linear operator satisfying $T \leq K$.

OR

2. (a) For any projection P on a Hilbert space H, prove the following
(i) $\langle Px, x\rangle = ||Px||^2$ (ii) $P \geq 0$ (iii) $||P|| \leq 1$; $||P|| = 1$ if $P(H) \neq \{0\}$.

(b) (i) If P_1 and P_2 are two projections on a Hilbert space H, then prove that $P = P_1P_2$ is a projection on H if and only if $P_1P_2 = P_2P_1$. In such a case prove that P projects H onto $Y = Y_1 \cap Y_2$, where $Y_j = P_j(H); j = 1, 2$ (ii) Prove that two closed subspaces Y and V of H are orthogonal if and only if their corresponding projections satisfy $P_Y P_V = 0$.

(10 x 4 = 40)