Reg. No…………………………………. Name……………………………………………

M.SC DEGREE END SEMESTER EXAMINATION NOVEMBER 2016

SEMESTER – 1: MATHEMATICS

COURSE: 16P1MATT01 –: LINEAR ALGEBRA

Time: Three Hours Max. Marks: 75

PART A

(Answer **all** questions. Each question carries **1.5** mark)

1. Check whether the set of all functions *f* such that $f(x^2) = ¿$ is a subspace of the vector space of all functions from ***R*** into ***R***.
2. Find a basis for the space of all 2×2 matrices with complex entries satisfying $A_{11}+A_{22}$ = 0.
3. Let *V* and *W* be vector spaces over a field F. Prove or disprove: every bijection from *V* into *W* is a linear transformation from *V* into *W*.
4. Prove that every *m*×*n* matrix over a field *F* defines a linear transformation from F^n into F^m.
5. *A* is a 3×3 with all its eigen values are integers. If determinant of *A* is -1 and one of the eigen values is 1, find the other eigen values.
6. What is a linear functional? Give an example.
7. If the characteristic polynomial of an operator is x^4-2x^2+1, what are the possible candidates for its minimal polynomial. Justify.
8. Let T be the linear operator on $\boldsymbol{R}^2$ defined by T(1,0) = (0,1) and T(0,1) = (-1 , 0). Find the subspace of $\boldsymbol{R}^2$ which is invariant under T.
9. If E is a projection of a vector space *V* and α϶ *V*, show that α – Eα is in the null space of E.
10. Check whether T: $\boldsymbol{R}^2 \to \boldsymbol{R}^2$ defined by T(*X*) = *AX* where A = $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is diagonalizable.

(1.5 x 10 = 15)

PART B

(Answer **any four** questions. Each question carries **5** marks.)

11. Let *V* be a vector spaces over the field *F*. Suppose there are a finite number of vectors in V which span *V*. Prove that *V* is finite dimensional.
12. Let *V* be an *n* – dimensional vector space over the field *F* and *W* be an *m* – dimensional vector space over *F*. Let Ɓ and Ɓ’ be ordered bases for *V* and *W* respectively. For any linear transformation T from *V* into *W* , prove that there is an *m*×*n* matrix *A* with entries in *F* such that $[T\alpha]_{B'} = A[\alpha]_B$

13. Define hyperspace. Let V be a finite dimensional vector space over the field F. Show that the null space of any nonzero linear functional on V is a hyperspace.
14. Let A be an $n\times n$ matrix with λ as an eigen value show that: (1) $k+\lambda$ is an eigen value of $A + kI$. (2) If A is nonsingular, $\frac{1}{\lambda}$ is an eigen value of A^{-1}.
15. Let V be a vector spaces and E be a projection on V. Show that V is the direct sum of R and N where R is the range space and N is the null space of E.

16. Let V be a finite dimensional vector space over the field F and let T be a linear operator on V. Prove that T is diagonalizable if and only if the minimal polynomial for T is of the form
$p = (x - c_1)(x - c_2)\ldots(x - c_k)$ where $c_i \in F$ are distinct.
(5 x 4 = 20)

PART C

(Answer **(a) or (b)** from each question. Each question carries **10** marks.)

17. (a) Let A and B be $m\times n$ matrices over the field F. Prove that the following statements are equivalent:
(1) A and B are row – equivalent. (2) A and B have the same row space. (3) $B = PA$, where P is an invertible $m\times m$ matrix.
(b) Let V be the space of all polynomial functions from $\mathbf{R}$ into $\mathbf{R}$ of atmost degree 2. That is the space of all functions f of the form $f(x) = c_0 + c_1x + c_2x^2$. Let t be a fixed real number and define $g_1(x) = 1$, $g_2(x) = x + t$, $g_3(x) = (x + t)^2$. Prove that $B = \{ g_1, g_2, g_3\}$ is a basis for V. Find the coordinates of $f(x) = c_0 + c_1x + c_2x^2$ in the basis B.

18. (a) (1) State and prove the Rank – Nullity theorem.
(2) Define $f: \mathbf{R}^3 \to \mathbf{R}^2$ by $f(1,0,0) = (1,-1)$ and $f(0,1,0) = (2,-2)$. Can f be a linear transformation? If so how many such linear transformations are there? Justify.
(b) (1) Let $g, f_1,\ldots f_r$ be linear functionals on a vector space V with respective null spaces $N, N_1,\ldots N_r$. Then prove that g is a linear combination of $f_1,\ldots f_r$ if and only if N contains the intersection $N_1 \cap \ldots \cap N_r$.
(2) Let n be a positive integer and F a field. Let W be the set of all vectors $(x_1,\ldots x_n)$ in F^n such that $x_1 + \ldots + x_n = 0$. Prove that W^0 consists of all linear functionals of the form $f(x_1,\ldots x_n) = c\sum_{j=1}^{n} x_j$.

19.(a) (1) Find the determinant of A^{10} where $A = \begin{bmatrix} 1 & 2 & 5 \\ 0 & -1 & -25 \\ 0 & 0 & 1 \end{bmatrix}$. Justify your answer.
(2) Let T be a linear operator on a finite dimensional vector space V. Let $c_1,\ldots c_k$ be the distinct characteristic values of T and let W_i be the null space of T -

c_iI. Prove that the following are equivalent: (1) T is diagonalizable. (2) The characteristic polynomial for T

is $f = (x - c_1)^{d_1} \ldots (x - c_k)^{d_k}$ and $\dim W_i = d_i$, i = 1,2,..k (3) $\sum_{i=1}^{k} \dim W_i = \dim V$.

(b) Diagonalize the matrix $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$.

20. (a) State and prove a necessary and sufficient condition for a linear operator on a finite dimensional vector space to be triangulable.

(b) Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 2 & -2 & 2 \\ 2 & -3 & 2 \end{bmatrix}$. Is A similar over the field of real numbers to a triangular matrix?

(10 x 4 = 40)

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