Reg. No............................. Name: .... **Q. Code: P103**

M SC DEGREE END SEMESTER EXAMINATION 2014 -15

SEMESTER -1: MATHEMATICS

COURSE CODE: P1MATT01:TITLE:LINEAR ALGEBRA

Time: 3 Hours Max. Marks: 75

Part A

Answer **Any Five** Each Question has 2 Marks

1. Let **V** be the set of pairs (x, y) of real numbers and let **F** be the field of real numbers.
   Define $(x_1, x_2)+(y_1, y_2)=(x_1+y_1, 0)$
   $c(x_1, x_2)=(c\,x_1, 0)$. Is **V**, with these operations a vector space over **F**?
2. Define $T: R^2 \to R^2$ by $T(x, y)=(1+x, y)$. Is **T** a linear transformation?
3. Define a non-singular linear transformation. Show that $T: R^2 \to R^2$ defined by $T(x, y)=(x+y, y)$ is non-singular.
4. Show that a linear transformation $T: R^n \to R^n$ is one-to-one if and only if it is onto.
5. Let **K** be a commutative ring with identity , and let **D** be a 2-linear function with the property that
   $D(A)=0$ for all 2×2 matrices **A** over **K** having equal rows. Show that **D** is alternating.
6. Let **T** be a linear operator on a finite dimensional vector space **V** and let **c** be a scalar. Show that the following are equivalent.
   (i) **c** is a characteristic value of **T**.
   (ii) The operator (**T-cI**) is invertible, where **I** is the identity operator on **V**.
7. Let **A** be an $n\times n$ triangular matrix over the field **F**. Prove that the characteristic values of **A** are the diagonal entries of **A**.
8. Define minimal polynomial for a linear operator **T** on a finite dimensional vector space **V**. Explain three properties which characterize minimal polynomial.

Part B

Answer **any Five** Each Question has 5 Marks

9. Let $\mathbf{W_1}$ and $\mathbf{W_2}$ be two subspaces of the vector space **V**. Show that $\mathbf{W_1} \cup \mathbf{W_2}$ is a subspace of **V** if and only if one is contained in the other.
10. Let **V** and **W** be finite dimensional vector spaces over the field **F**. Prove that **V** and **W** are isomorphic if and only if dim**V** = dim**W**.
11. If **f** and **g** are linear functionals on a vector space **V**, then show that **g** = **cf** for some scalar **c** if and only if the null space of **g** contains the null space of **f**.
12. Let $T: R^2 \to R^2$ be defined by $T(x, y)=(-y, x)$.
    (i) What is the matrix of **T** in the standard ordered basis for R^2.
    (ii) What is the matrix of **T** in the ordered basis $B=\{(1,2),(1,-1)\}$.
13. Let **K** be a commutative ring with identity, and let **A** and **B** be $n\times n$ matrices over **K**. If det (**A**) denotes the determinant of **A**, show that det (**AB**) = det (**A**).det (**B**).
14. Let **A** and **B** be $n\times n$ matrices over the field **F**. Prove that if (**I-AB**) is invertible, then (**I-BA**) is also invertible and $(\mathbf{I-BA})^{-1} = \mathbf{I} + \mathbf{B}\,(\mathbf{I-AB})^{-1}\mathbf{A}$.

15. Let **T** be a linear operator on the n-dimensional vector space **V**, and suppose that **T** has n distinct characteristic values. Prove that **T** is diagonalizable.
16. Let **T** be a linear operator on the n-dimensional vector space **V**. Show that the characteristic and minimal polynomials for **T** have the same roots, except for multiplicities.

Part C

Answer **(a) or (b) from Each Questions**. Each Question has 10 Marks

17. (a) Let **V** be an n-dimensional vector space over the field **F**, and let $\mathbf{B_1}$ and $\mathbf{B_2}$ be two ordered bases of **V**. Show that there exists unique, invertible, $n \times n$ matrix **P** with entries in **F** suchthat
    (i) $[x]_{B_1} = P[x]_{B_2}$
    (ii) $[x]_{B_2} = P^{-1}[x]_{B_1}$
    (b)(i) Let **V** be the set of all real numbers. Regard **V** as a vector space over the field of rational numbers, with usual operations. Prove that **V** is not finite dimensional.
    (ii) Let **V** be the vector space of all 2×2 matrices over the field **F**. Prove that **V** has dimension 4 by exhibiting a basis for **V**.
18. (a) Let **V** be a finite dimensional vector space over the field **F**, and let $\{x_1, x_2, \ldots x_n\}$ be an ordered basis for **V**. Let **W** be a vector space over the same field **F** and $\{y_1, y_2, \ldots y_n\}$ be any set of vectors in **W**. Then show that there is precisely one linear transformation **T** from **V** into **W** such that $T(x_j) = y_j$ for $j = 1, 2, \ldots n$.
    (b) Let **V** be the vector space of all polynomial functions **f: R→ R** of the form $f(x) = a + bx + cx^2 + dx^3$; a, b, c, d real numbers. Define the map **D: V→ V** by **D** $(a + bx + cx^2 + dx^3) = b + 2cx + 3dx^2$. Prove that **D** is a linear transformation and find the matrix of **D** with respect to the ordered basis **B**= { f_0, f_1, f_2, f_3} where $f_j(x) = x^j, j = 0,1,2,3.$
19. (a) Let **A** be an $n \times n$ matrix over the field **F**. Then show that **A** is invertible over **F** if and only if det(**A**)≠0. When **A** is invertible, show that $\mathbf{A^{-1}=det(A)^{-1}.adj(A)}$, where **adj(A)** is the adjoint of **A**.
    (b) (i) For an $n \times n$ matrix **A** with complex entries, if $\mathbf{A^t}$ denotes the transpose of **A** then prove that $\mathbf{det(A^t) = det(A)}$.
    (ii) If **A** is a skew symmetric $n \times n$ matrix with complex entries (that is $\mathbf{A^t = -A}$) and n is odd, prove that det(**A**)=0.
    (iii) If **A** is an orthogonal $n \times n$ matrix with complex entries (that is $\mathbf{AA^t = I}$), prove that det(**A**)=±1.
20. (a) Let **T** be a linear operator on the finite dimensional vector space **V**. Let $\mathbf{c_1}, \mathbf{c_2}, \ldots \mathbf{c_k}$ be the distinct characteristic values of **T** and let $\mathbf{W_i}$ be the characteristic space associated with the characteristic value $\mathbf{c_i}$. If $\mathbf{W = W_1 + W_2 + \ldots + W_k}$ then show that
    dim**W**=dim $\mathbf{W_1}$+dim$\mathbf{W_2}$+… +dim$\mathbf{W_k}$.
    (b)(i) If **E** is a projection on a vector space **V** with range **R** and null space **N**, show that $V = R \oplus N$.
    (ii) Show that a vector x is in the range **R** of **E** if and only if $E(x) = x$.
    (iii) Find a projection **E** which projects $\mathbf{R^2}$ on to the subspace spanned by (1,-1) along the subspace spanned by (1, 2).

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